

The Weitzenböck formula on the Wiener space and its application to the asymptotic estimate of entropy

Dejun Luo*

Key Lab of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Abstract

We consider the Fokker-Planck equation on the abstract Wiener space associated to the Ornstein-Uhlenbeck operator. Using the Weitzenböck formula, we prove an explicit estimate on the time derivative of the entropy of the solution to the Fokker-Planck equation.

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1 Introduction

The second law in thermodynamics asserts that the entropy of an isolated physical system always increases; an example is the entropy of the solution to the heat equation on a Riemannian manifold. Recently motivated by the ground-breaking work of Perelman, there have been intensive studies on the entropy (i.e. Perelman's W -functional) of the solution to the heat equation. In [10, 11], the author proved the monotonicity of the entropy and showed its connection with the geometry of the manifold. In the recent work [6], we presented several estimates on the time derivative of entropy of the solution to the heat equation on a Riemannian manifold, in terms of the lower bound on the Ricci curvature and the first eigenvalue of the Laplacian operator. In the present work we intend to generalize these results to the infinite dimensional case.

Let (W, H, μ) be an abstract Wiener space (see the beginning of Section 2 for its definition) and \mathcal{L} the Ornstein-Uhlenbeck operator on W . Consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} u_t = \mathcal{L} u_t, \quad u|_{t=0} = u_0, \quad (1.1)$$

where $u_0 \in L^p(W)$ for some $p > 1$. This equation is understood in the weak sense, see (3.1). As shown in Section 3, equation (1.1) has a unique solution which is given by $u_t = P_t u_0$ with P_t being the Ornstein-Uhlenbeck semigroup on W . Moreover u_t also solves (1.1) in the Fréchet sense. Now suppose that $u_0 > 0$; then $u_t > 0$ and $\int_W u_t d\mu \equiv \int_W u_0 d\mu$ for all $t \geq 0$. Define the entropy by

$$\text{Ent}(u_t) = - \int_W u_t \log u_t d\mu.$$

From the simple inequality $-x \log x \leq 1 - x$ for all $x \geq 0$, we know that $\text{Ent}(u_t) \leq 1 - \int_W u_0 d\mu$ for any $t > 0$. The formal calculation gives us

$$\frac{d}{dt} \text{Ent}(u_t) = - \int_W \left((\log u_t) \frac{\partial}{\partial t} u_t + \frac{\partial}{\partial t} u_t \right) d\mu$$

*Email: luodj@amss.ac.cn

$$= - \int_W (\log u_t) \mathcal{L} u_t \, d\mu = \int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu,$$

where the last equality follows from the integration by parts formula. Therefore the entropy $\text{Ent}(u_t)$ is an increasing function of t if the initial condition u_0 is not a constant. As in [6], we will estimate the rate of change of the entropy as $t \rightarrow \infty$, by making use of the Weitzenböck formula on W .

Denote by $\mathbb{D}_1^p(W)$ the first order Sobolev space on the Wiener space W . The main result of this paper is

Theorem 1.1. *Let $p > 1$. Suppose $u_0 \in \mathbb{D}_1^{4p}(W)$ such that $u_0 \geq \varepsilon_0$ for some positive constant $\varepsilon_0 > 0$. Then*

$$\frac{d}{dt} \text{Ent}(u_t) \leq e^{-2t} \int_W \frac{|\nabla u_0|_H^2}{u_0} \, d\mu.$$

This theorem will be proved in Section 3. The above estimate is consistent with the result in [6, Example 2.4]. Indeed, let \mathcal{L}_n be the n -dimensional version of the Ornstein-Uhlenbeck operator \mathcal{L} (see (2.4) for its definition); then we have $\mathcal{L}_n = 2L$ where L is defined in [6, Example 2.4] with $k = 1$. Slight modification of the arguments in [6, Example 2.4] will give us that the time derivative of the entropy of the transition density is e^{-2t} . We would like to mention that, by [3, Theorem 1.5] and the metric measure theory (see e.g. [7, 14, 15]), the lower bound for the Ricci curvature of the Wiener space W is 1, hence the above theorem is also in accordance with the main result in [6] if we consider the Laplacian operator Δ instead of $\frac{1}{2}\Delta$.

Compared to [6, Theorem 1.1], the main difficulties in the infinite dimensional situation are: (1) the justification of that various functionals belong to the domain of the Ornstein-Uhlenbeck operator \mathcal{L} , and (2) the validness of the differentiation under the integral sign. Fortunately the unique solution u_t to the Fokker-Planck equation (1.1) is sufficiently regular, and the equation can actually be understood in the sense of Fréchet differential. These observations make our computations possible. To avoid the technical difficulties, we assume that the initial value u_0 has a positive lower bound $\varepsilon_0 > 0$, so that the estimations become easier.

The paper is organized as follows. We recall in Section 2 some preliminary elements in the Malliavin calculus and prove the Weitzenböck formula on W associated to the Ornstein-Uhlenbeck operator \mathcal{L} (cf. Theorem 2.2). In Section 3 we first show that $u_t = P_t u_0$ is the unique solution to the equation (1.1); after that we establish an equality which is essential for proving the main result of this paper, see Theorem 3.4. Then by following the idea in [6], we present the proof of Theorem 1.1. Finally in the Appendix, we give the proof of a result concerning the differentiation under the integral sign which is needed in the proof of the main theorem.

2 Preliminaries in Malliavin calculus and the Weitzenböck formula

In this section we recall some basic facts in the Malliavin calculus and present a Weitzenböck type formula on the Wiener space associated to the Ornstein-Uhlenbeck operator. Let (W, H, μ) be an abstract Wiener space in the sense of L. Gross, i.e. W is a separable Banach space, H is a separable Hilbert space and μ is a Borel probability on W , such that H is continuously and densely embedded into W and for any $\ell \in W^*$ (the dual space of W), we have

$$\int_W e^{\sqrt{-1}\ell(w)} \, d\mu(w) = e^{-|\ell|_H^2/2},$$

where $|\cdot|_H$ is the norm in H associated to the inner product $\langle \cdot, \cdot \rangle_H$. In the following, we fix an orthonormal basis $\{h_i : i \geq 1\}$ of H , with $h_i \in W^*$ for all $i \geq 1$. Define $H_n = \text{span}\{h_i : 1 \leq i \leq n\}$ and

$$\pi_n(w) = \sum_{i=1}^n h_i(w) h_i, \quad w \in W.$$

Then π_n is a continuous linear map from W onto H_n , and the restriction of $\pi_n|_H$ is the orthogonal projection. It is known that the push forward $\mu_n := (\pi_n)_\# \mu$ is the standard Gaussian measure on the n -dimensional Euclidean space H_n .

We refer to [2, 4, 9, 12] for the background in Malliavin calculus. Let K be a separable Hilbert space. Denote by $K \otimes H$ the Hilbert space of Hilbert-Schmidt operators L from K to H , and $\|L\|_{K \otimes H}$ the Hilbert-Schmidt norm. For some $p > 1$ and $Z \in L^p(W, K)$, we say that $Z \in \mathbb{D}_1^p(W, K)$ if there exists $\nabla Z \in L^p(W, H \otimes K)$ such that for each $h \in H$,

$$\langle \nabla Z, h \rangle_H = D_h Z = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} Z(w + \varepsilon h) \quad \text{holds in } L^{p-}.$$

The space $\mathbb{D}_1^p(W, K)$ is complete under the norm:

$$\|Z\|_{\mathbb{D}_1^p(W, K)} = (\|Z\|_{L^p(W, K)}^p + \|\nabla Z\|_{L^p(W, H \otimes K)}^p)^{1/p}.$$

In the same way we can define Sobolev spaces $\mathbb{D}_m^p(W, K)$ of higher orders $m \geq 1$. Notice that $\mathbb{D}_0^p(W, K) = L^p(W, K)$. A K -valued functional Z is called cylindrical if there exist $N, M \geq 1$, $f_i \in C_b^\infty(\mathbb{R}^M)$ and $k_i \in K$ ($1 \leq i \leq N$), such that

$$Z = \sum_{i=1}^N f_i(h_1(w), \dots, h_M(w)) k_i.$$

By the Schmidt orthogonalization procedure, we may always assume that $\{k_1, \dots, k_N\}$ is an orthonormal family. Note that $Z : W \rightarrow K$ is Fréchet differentiable of any order. We denote by $\text{Cylin}(W, K)$ the space of K -valued cylindrical functionals, which is dense in $\mathbb{D}_m^p(W, K)$. If $K = \mathbb{R}$, we simply write $\mathbb{D}_m^p(W)$ and $\text{Cylin}(W)$. A basic result in Malliavin calculus is that the divergence $\delta(Z) \in \mathbb{D}_{m-1}^p(W, K)$ exists for $Z \in \mathbb{D}_m^p(W, H \otimes K)$ (see [12, Proposition 1.5.7]), and there is $C_{p,m} > 0$ such that

$$\|\delta(Z)\|_{\mathbb{D}_{m-1}^p(W, K)} \leq C_{p,m} \|Z\|_{\mathbb{D}_m^p(W, H \otimes K)}. \quad (2.1)$$

The Mehler formula below defines the Ornstein-Uhlenbeck semigroup P_t on W :

$$P_t F(x) = \int_W F(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\mu(y). \quad (2.2)$$

Here are some basic properties of P_t that will be used later.

Proposition 2.1. (1) For any $t > 0$, $P_t(\text{Cylin}(W)) \subset \text{Cylin}(W)$.

(2) For any $t > 0$ and $p \in [1, +\infty]$, we have for all $u \in L^p(W)$, $\|P_t u\|_{L^p} \leq \|u\|_{L^p}$ and $\lim_{t \rightarrow 0} \|P_t u - u\|_{L^p} = 0$.

(3) P_ε is self-adjoint in $L^2(W)$. Furthermore, for any $p \in (1, +\infty)$ and $u \in L^p(W)$, $v \in L^q(W)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_W u P_\varepsilon v d\mu = \int_W v P_\varepsilon u d\mu.$$

- (4) For every $t > 0$, $p > 1$ and $m \geq 1$, we have $P_t u \in \mathbb{D}_m^p(W)$ for any $u \in L^p(W)$, and there is $C_{p,m} > 0$ such that

$$\|\nabla^m P_t u\|_{L^p(W, H^{\otimes m})} \leq C_{p,m} A_t^m \|u\|_{L^p(W)},$$

where $H^{\otimes m} = \underbrace{H \otimes \cdots \otimes H}_{m \text{ times}}$ and $A_t = e^{-t}/\sqrt{1 - e^{-2t}}$.

Recall that the last result is due to Sugita [16] (see also [2, Exercise 6.8]). The infinitesimal generator \mathcal{L} of P_t is called the Ornstein-Uhlenbeck operator:

$$\frac{\partial}{\partial t} P_t F = \mathcal{L} P_t F = P_t \mathcal{L} F, \quad F \in \text{Cylin}(W). \quad (2.3)$$

It is known that $\mathcal{L} = -\delta \circ \nabla$, hence \mathcal{L} is a continuous operator from $\mathbb{D}_{m+2}^p(W)$ to $\mathbb{D}_m^p(W)$ for all $m \geq 0$ and $p > 1$. Furthermore, if $F \in \text{Cylin}(W)$ has the expression $F(w) = f \circ \pi_n(w)$ for some $f \in C_b^\infty(H_n)$, then

$$\mathcal{L} F(w) = (\mathcal{L}_n f) \circ \pi_n(w), \quad (2.4)$$

where \mathcal{L}_n is the Ornstein-Uhlenbeck operator on H_n : $\mathcal{L}_n f(x) = \sum_{i=1}^n (\partial_{ii} f(x) - x^i \partial_i f(x))$. Our main result of this section is

Theorem 2.2 (Weitzenböck formula). *Let $F \in \mathbb{D}_3^{2p}(W)$ for some $p > 1$. Then*

$$\mathcal{L}(|\nabla F|_H^2) = 2\langle \nabla F, \nabla \mathcal{L} F \rangle_H + 2|\nabla F|_H^2 + 2\|\nabla^2 F\|_{H \otimes H}^2. \quad (2.5)$$

An integral form of the Weitzenböck formula is given in [2, Section 5.2], which is the interpretation for 1-forms of the de Rham-Hodge-Kodaira decomposition in [13]. The equality (2.5) can be proved by following some of the arguments in [2, Section 5.3]. For the readers' convenience, we include its proof here. First we consider the case where F is cylindrical.

Lemma 2.3. *The equality (2.5) holds for all $F \in \text{Cylin}(W)$.*

Proof. Let $F(w) = f \circ \pi_n(w)$ for some smooth function $f : H_n \rightarrow \mathbb{R}$. We have $\nabla F(w) = \sum_{i=1}^n (\partial_i f)(\pi_n(w)) h_i$, hence

$$|\nabla F(w)|_H^2 = \sum_{i=1}^n [(\partial_i f)(\pi_n(w))]^2 = |\nabla_n f|^2(\pi_n(w)), \quad (2.6)$$

where ∇_n is the gradient on the Euclidean space H_n . By (2.4), we obtain

$$\mathcal{L}(|\nabla F|_H^2)(w) = [\mathcal{L}_n(|\nabla_n f|^2)](\pi_n(w)). \quad (2.7)$$

Now direct computations lead to

$$\mathcal{L}_n(|\nabla_n f|^2)(x) = 2\|\text{Hess} f\|_{H_n \otimes H_n}^2 + 2\langle \nabla_n f, \nabla_n \Delta_n f \rangle - 2(\text{Hess} f)(x, \nabla_n f), \quad x \in \mathbb{R}^n,$$

where Δ_n is the Laplacian on H_n . Notice that $\nabla_n \langle x, \nabla_n f \rangle = \nabla_n f + (\text{Hess} f) \cdot x$, therefore

$$(\text{Hess} f)(x, \nabla_n f) = \langle \nabla_n \langle x, \nabla_n f \rangle, \nabla_n f \rangle - |\nabla_n f|^2.$$

Substituting this into the above equality and by the definition of \mathcal{L}_n , we get

$$\begin{aligned} \mathcal{L}_n(|\nabla_n f|^2)(x) &= 2\|\text{Hess} f\|_{H_n \otimes H_n}^2 + 2\langle \nabla_n f, \nabla_n \Delta_n f \rangle - 2\langle \nabla_n \langle x, \nabla_n f \rangle, \nabla_n f \rangle + 2|\nabla_n f|^2 \\ &= 2\|\text{Hess} f\|_{H_n \otimes H_n}^2 + 2\langle \nabla_n f, \nabla_n \mathcal{L}_n f \rangle + 2|\nabla_n f|^2. \end{aligned}$$

Combining this with (2.4), (2.6) and (2.7), we get the desired result. \square

Proof of Theorem 2.2. Since $\text{Cylin}(W)$ is dense in $\mathbb{D}_3^{2p}(W)$, there exists a sequence $\{F_n : n \geq 1\}$ of cylindrical functionals such that $\lim_{n \rightarrow \infty} \|F_n - F\|_{\mathbb{D}_3^{2p}(W)} = 0$. By Lemma 2.3, for all $n \geq 1$,

$$\mathcal{L}(|\nabla F_n|_H^2) = 2\langle \nabla F_n, \nabla \mathcal{L}F_n \rangle_H + 2|\nabla F_n|_H^2 + 2\|\nabla^2 F_n\|_{H \otimes H}^2. \quad (2.8)$$

It remains to show that both sides of the above equality converge in $L^p(W)$. First it is clear that $|\nabla F_n|_H^2$ converges to $|\nabla F|_H^2$ in $\mathbb{D}_2^p(W)$. Note that the Ornstein-Uhlenbeck operator is a continuous map from $\mathbb{D}_2^p(W)$ to $L^p(W)$, thus

$$\lim_{n \rightarrow \infty} \|\mathcal{L}(|\nabla F_n|_H^2) - \mathcal{L}(|\nabla F|_H^2)\|_{L^p} = 0. \quad (2.9)$$

Next by the triangular inequality,

$$\begin{aligned} \|\langle \nabla F_n, \nabla \mathcal{L}F_n \rangle_H - \langle \nabla F, \nabla \mathcal{L}F \rangle_H\|_{L^p} &\leq \|\langle \nabla F_n - \nabla F, \nabla \mathcal{L}F_n \rangle_H\|_{L^p} \\ &\quad + \|\langle \nabla F, \nabla \mathcal{L}F_n - \nabla \mathcal{L}F \rangle_H\|_{L^p}. \end{aligned}$$

Cauchy's inequality leads to

$$\begin{aligned} \|\langle \nabla F_n - \nabla F, \nabla \mathcal{L}F_n \rangle_H\|_{L^p} &\leq \|\nabla F_n - \nabla F\|_{L^{2p}(W, H)} \|\nabla \mathcal{L}F_n\|_{L^{2p}(W, H)} \\ &\leq \|F_n - F\|_{\mathbb{D}_1^{2p}(W)} \cdot C_p \|F_n\|_{\mathbb{D}_3^{2p}(W)}, \end{aligned} \quad (2.10)$$

where the last inequality follows from the boundedness of the operator $\mathcal{L} : \mathbb{D}_3^{2p}(W) \rightarrow \mathbb{D}_1^{2p}(W)$. Since the sequence $\|F_n\|_{\mathbb{D}_3^{2p}(W)}$ is bounded, we arrive at

$$\lim_{n \rightarrow \infty} \|\langle \nabla F_n - \nabla F, \nabla \mathcal{L}F_n \rangle_H\|_{L^p} = 0.$$

Similarly we have

$$\|\langle \nabla F, \nabla \mathcal{L}F_n - \nabla \mathcal{L}F \rangle_H\|_{L^p} \leq C_p \|F\|_{\mathbb{D}_1^{2p}(W)} \|F_n - F\|_{\mathbb{D}_3^{2p}(W)}$$

whose right hand side tends to 0 as $n \rightarrow \infty$. From this and (2.10), we conclude that the first term on the right hand side of (2.8) converges in $L^p(W)$ to $2\langle \nabla F, \nabla \mathcal{L}F \rangle_H$. Finally it is easy to show that the last two terms of equality (2.8) also converge in $L^p(W)$ to $2|\nabla F|_H^2$ and $2\|\nabla^2 F\|_{H \otimes H}^2$, respectively. Combining these results with (2.9), we complete the proof of Theorem 2.2 by taking limit in (2.8). \square

3 Entropy of the solution to the Fokker-Planck equation on W

In this section we will estimate the time derivative of entropy of the solution to the Fokker-Planck equation associated to the Ornstein-Uhlenbeck operator. From (2.3), we see that for any initial value $u_0 \in \text{Cylin}(W)$, $u_t := P_t u_0$ gives the solution to the Fokker-Planck equation in the classical sense. In the following we show that this fact remains true for all $u_0 \in L^p(W)$, where $p > 1$.

First we introduce the notion of weak solution to (1.1). A function $u \in L^\infty([0, \infty), L^p(W))$ is called a weak solution to the equation (1.1) if for any $\alpha \in C_c^\infty([0, \infty))$ and $F \in \text{Cylin}(W)$, it holds

$$-\alpha(0) \int_W F u_0 \, d\mu = \int_0^\infty \int_W [\alpha'(t)F + \alpha(t)\mathcal{L}F] u_t \, d\mu dt. \quad (3.1)$$

For $u_0 \geq 0$ satisfying $\int_W u_0 d\mu = 1$ and $-\text{Ent}(u_0) = \int_W u_0 \log u_0 d\mu < +\infty$, it is shown in [3, Theorem 3.7] that the weak solution u_t of equation (1.1) can be constructed via De Giorgi's "minimizing movement" approximation. But the uniqueness of solutions is not considered there. The existence and uniqueness of general Fokker-Planck type equations are studied in [8]; however, those results do not apply to the equation (3.1) (see [8, Remark 4.6]).

We first prove the following simple result. For $n \geq 1$, let \mathbb{E}^{H_n} be the conditional expectation on W with respect to the σ -field generated by cylindrical functionals of the form $F = f \circ \pi_n$. We also denote by \mathbb{E}^μ and \mathbb{E}^{μ_n} the expectations on W and H_n respectively.

Lemma 3.1. *Let $u_0 \in L^p(W)$ and $u_t = P_t u_0$, $t \geq 0$. For $n \geq 1$, define the function $u_n(t) \in L^p(H_n, \mu_n)$ such that $u_n(t) \circ \pi_n = \mathbb{E}^{H_n}(u_t)$. Then*

$$u_n(t) = P_t^{(n)} u_n(0),$$

where $P_t^{(n)}$ is the Ornstein-Uhlenbeck semigroup on H_n .

Proof. For any $F = f \circ \pi_n$ with $f \in C_b^\infty(H_n)$, by the symmetry of the operator P_t (see Proposition 2.1(3)), we have

$$\mathbb{E}^{\mu_n}[f u_n(t)] = \mathbb{E}^\mu[(f \circ \pi_n) \mathbb{E}^{H_n}(u_t)] = \mathbb{E}^\mu[F u_t] = \mathbb{E}^\mu[(P_t F) u_0].$$

By the Mehler formula (2.2), it is clear that $P_t F = (P_t^{(n)} f) \circ \pi_n$, hence

$$\begin{aligned} \mathbb{E}^{\mu_n}[f u_n(t)] &= \mathbb{E}^\mu[(P_t^{(n)} f) \circ \pi_n \cdot u_0] = \mathbb{E}^\mu[(P_t^{(n)} f) \circ \pi_n \cdot \mathbb{E}^{H_n}(u_0)] \\ &= \mathbb{E}^{\mu_n}[(P_t^{(n)} f) \cdot u_n(0)] = \mathbb{E}^{\mu_n}[f P_t^{(n)} u_n(0)], \end{aligned}$$

where the last equality follows from the symmetry of $P_t^{(n)}$. Since $f \in C_b^\infty(H_n)$ is arbitrary, we complete the proof. \square

Theorem 3.2. *Let $u_0 \in L^p(W)$ with $p > 1$. Then $u_t := P_t u_0$ is the unique weak solution to the Fokker-Planck equation (1.1) in the space*

$$\mathcal{S} = \{v \in L^\infty([0, \infty), L^p(W)) : \nabla v \in L^1([0, \infty), L^p(W, H))\}.$$

Proof. First we check that $u_t = P_t u_0$ is really a solution of (1.1). Letting $m = 1$ in Proposition 2.1(4) and by the definition of A_t , it is clear that $\nabla(P_t u_0) \in L^1([0, \infty), L^p(W, H))$, hence $u \in \mathcal{S}$. Next we take a sequence $\{u^{(n)} : n \geq 1\}$ of cylindrical functionals such that $\lim_{n \rightarrow \infty} \|u^{(n)} - u_0\|_{L^p} = 0$. For any $n \geq 1$, let $u_t^{(n)} = P_t u^{(n)}$. Then by (2.3), we have $\frac{\partial}{\partial t} u_t^{(n)} = \mathcal{L} u_t^{(n)}$. Thus for any $\alpha \in C_c^\infty([0, \infty))$ and $F \in \text{Cylin}(W)$, it holds

$$-\alpha(0) \int_W F u^{(n)} d\mu = \int_0^\infty \int_W [\alpha'(t) F + \alpha(t) \mathcal{L} F] u_t^{(n)} d\mu dt. \quad (3.2)$$

By Proposition 2.1(2), we have

$$\sup_{0 \leq t < \infty} \|u_t^{(n)} - u_t\|_{L^p} \leq \|u^{(n)} - u_0\|_{L^p} \rightarrow 0$$

as $n \rightarrow \infty$. Noting that $\mathcal{L} F \in L^q(W)$ where q is the conjugate number of p : $p^{-1} + q^{-1} = 1$, taking limit in (3.2) gives us the equality (3.1). That is to say, $u_t = P_t u_0$ is a weak solution to the Fokker-Planck equation (1.1).

Next we prove the uniqueness of weak solutions to (3.1). Let $v \in \mathcal{S}$ be any solution of (3.1) with $v_0 = u_0$. For any $n \geq 1$, define $u_n \in L^p(H_n, \mu_n)$ such that

$$u_n \circ \pi_n = \mathbb{E}^{H_n}(u_0).$$

Similarly we define $v_n(t) \in L^p(H_n, \mu_n)$ from v_t . It is easy to know that v_n belongs to the space

$$\mathcal{S}_n = \{v_n \in L^\infty([0, \infty), L^p(H_n, \mu_n)) : \nabla_n v_n \in L^1([0, \infty), L^p(H_n, \mu_n))\}.$$

Now fix $n \geq 1$. Then for any $F = f \circ \pi_n \in \text{Cylin}(W)$ and $\alpha \in C_c^\infty([0, \infty))$, it holds

$$-\alpha(0) \int_W F u_0 \, d\mu = \int_0^\infty \int_W [\alpha'(t)F + \alpha(t)\mathcal{L}F] v_t \, d\mu dt. \quad (3.3)$$

We have

$$\begin{aligned} \int_W F u_0 \, d\mu &= \mathbb{E}^\mu[(f \circ \pi_n)u_0] = \mathbb{E}^\mu[(f \circ \pi_n)\mathbb{E}^{H_n}(u_0)] \\ &= \mathbb{E}^\mu[(f \circ \pi_n)(u_n \circ \pi_n)] = \int_{H_n} f u_n \, d\mu_n. \end{aligned}$$

Now for any $t \in [0, T]$, in the same way we have

$$\int_W F v_t \, d\mu = \int_{H_n} f v_n(t) \, d\mu_n$$

and by (2.4),

$$\int_W (\mathcal{L}F) v_t \, d\mu = \int_{H_n} (\mathcal{L}_n f) v_n(t) \, d\mu_n.$$

Therefore the equation (3.3) becomes

$$-\alpha(0) \int_{H_n} f u_n \, d\mu_n = \int_0^\infty \int_{H_n} [\alpha'(t)f + \alpha(t)\mathcal{L}_n f] v_n(t) \, d\mu_n dt. \quad (3.4)$$

This means that v_n is a weak solution to the finite dimensional Fokker-Planck equation

$$\frac{\partial}{\partial t} v_n(t) = \mathcal{L}_n v_n(t)$$

with initial condition $v_n(0) = u_n$. Approximating u_n by smooth functions with respect to the norm in $L^p(H_n, \mu_n)$ and following the argument of the starting part of this theorem, we can show that $P_t^{(n)} u_n$ is also the solution to (3.4) with the same initial value u_n . Moreover $P_t^{(n)} u_n \in \mathcal{S}_n$. By the uniqueness of solutions in the finite dimensional case (cf. [8, Theorem 4.10] restricted to the finite dimensional context or [5, Corollary 1]), we obtain $v_n(t) = P_t^{(n)} u_n$ for all $t > 0$. Lemma 3.1 tells us that $v_n(t) \circ \pi_n = (P_t^{(n)} u_n) \circ \pi_n = \mathbb{E}^{H_n}(u_t)$. Letting n tend to infinity, we conclude that $v_t = u_t = P_t u_0$. Therefore we get the uniqueness of weak solutions to the Fokker-Planck equation (3.1). \square

In the next proposition we show that $u_t = P_t u_0$ is also a solution to equation (1.1) in the strong sense.

Proposition 3.3. *Let $u_0 \in L^p(W)$ for some $p > 1$. Then for any $m \geq 0$, the following equality holds in $\mathbb{D}_m^p(W)$:*

$$\frac{\partial}{\partial t} P_t u_0 = \mathcal{L}(P_t u_0), \quad \text{for all } t > 0.$$

Proof. By Proposition 2.1(4), for any $t > 0$,

$$\|P_t u_0\|_{\mathbb{D}_m^p(W)} \leq \left[\sum_{i=0}^m C_{p,i}^p A_t^{ip} \|u_0\|_{L^p}^p \right]^{1/p} \leq \bar{C}_{p,m} \|u_0\|_{L^p} \sum_{i=0}^m A_t^i. \quad (3.5)$$

Therefore by the boundedness of the Ornstein-Uhlenbeck operator $\mathcal{L} : \mathbb{D}_{m+2}^p(W) \rightarrow \mathbb{D}_m^p(W)$,

$$\|\mathcal{L} P_t u_0\|_{\mathbb{D}_m^p(W)} \leq \hat{C}_{p,m} \|P_t u_0\|_{\mathbb{D}_{m+2}^p(W)} \leq \tilde{C}_{p,m} \|u_0\|_{L^p} \sum_{i=0}^{m+2} A_t^i. \quad (3.6)$$

Recall that $A_t = e^{-t}/\sqrt{1-e^{-2t}}$. Thus for any $0 < s < t < \infty$, the right hand side is integrable on the interval $[s, t]$. In particular, (3.5) implies that the curve $(0, \infty) \ni t \rightarrow P_t u_0$ is locally integrable in the Sobolev space $\mathbb{D}_m^p(W)$.

Now we take a sequence $\{u_n : n \geq 1\}$ of cylindrical functionals such that $\|u_n - u_0\|_{L^p(W)} \rightarrow 0$ as n goes to ∞ . Then for any $t > 0$, (3.5) leads to

$$\overline{\lim}_{n \rightarrow \infty} \|P_t u_n - P_t u_0\|_{\mathbb{D}_m^p(W)} \leq \bar{C}_{p,m} \left(\sum_{i=0}^m A_t^i \right) \lim_{n \rightarrow \infty} \|u_n - u_0\|_{L^p} = 0. \quad (3.7)$$

Similarly, for all $0 < s < t < \infty$, by (3.6)

$$\begin{aligned} \left\| \int_s^t \mathcal{L} P_\tau u_n \, d\tau - \int_s^t \mathcal{L} P_\tau u_0 \, d\tau \right\|_{\mathbb{D}_m^p(W)} &\leq \int_s^t \|\mathcal{L} P_\tau u_n - \mathcal{L} P_\tau u_0\|_{\mathbb{D}_m^p(W)} \, d\tau \\ &\leq \tilde{C}_{p,m} \|u_n - u_0\|_{L^p} \sum_{i=0}^{m+2} \int_s^t A_\tau^i \, d\tau, \end{aligned} \quad (3.8)$$

whose right hand side tends to 0 as $n \rightarrow \infty$. Now by (2.3), for every $n \geq 1$, we have

$$P_t u_n - P_s u_n = \int_s^t \mathcal{L} P_\tau u_n \, d\tau.$$

With (3.7) and (3.8) in mind, letting $n \rightarrow \infty$ in the above equality gives us

$$P_t u_0 - P_s u_0 = \int_s^t \mathcal{L} P_\tau u_0 \, d\tau, \quad (3.9)$$

which holds in any Sobolev space $\mathbb{D}_m^p(W)$. In particular, by (3.6),

$$\|P_t u_0 - P_s u_0\|_{\mathbb{D}_m^p(W)} \leq \int_s^t \|\mathcal{L} P_\tau u_0\|_{\mathbb{D}_m^p(W)} \, d\tau \leq \tilde{C}_{p,m} \|u_0\|_{L^p} \sum_{i=0}^{m+2} \int_s^t A_\tau^i \, d\tau,$$

which implies that $(0, \infty) \ni t \mapsto P_t u_0 \in \mathbb{D}_m^p(W)$ is continuous for any $m \geq 1$.

Now by the boundedness of \mathcal{L} ,

$$\begin{aligned} \left\| \frac{P_t u_0 - P_s u_0}{t-s} - \mathcal{L} P_s u_0 \right\|_{\mathbb{D}_m^p(W)} &\leq \frac{1}{t-s} \int_s^t \|\mathcal{L} P_\tau u_0 - \mathcal{L} P_s u_0\|_{\mathbb{D}_m^p(W)} \, d\tau \\ &\leq \frac{C'_{p,m}}{t-s} \int_s^t \|P_\tau u_0 - P_s u_0\|_{\mathbb{D}_{m+2}^p(W)} \, d\tau. \end{aligned}$$

The continuity of $(0, \infty) \ni \tau \mapsto P_\tau u_0 \in \mathbb{D}_{m+2}^p(W)$ gives rise to

$$\lim_{t \rightarrow s} \left\| \frac{P_t u_0 - P_s u_0}{t - s} - \mathcal{L} P_s u_0 \right\|_{\mathbb{D}_m^p(W)} = 0.$$

The proof is complete. \square

Now we prove an equality which is critical in the proof of the main result.

Theorem 3.4. *Let $u_t = P_t u_0$ with $u_0 \in L^{4p}(W)$ and $u_0 \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. Then*

$$\left(\mathcal{L} - \frac{\partial}{\partial t} \right) \left(\frac{|\nabla u_t|_H^2}{u_t} \right) = \frac{2}{u_t} |\nabla u_t|_H^2 + \frac{2}{u_t} \left\| \nabla^2 u_t - \frac{\nabla u_t \otimes \nabla u_t}{u_t} \right\|_{H \otimes H}^2.$$

Proof. By Proposition 2.1(4), $u_t = P_t u_0 \in \mathbb{D}_3^{4p}(W)$, thus $|\nabla u_t|_H^2 \in \mathbb{D}_2^{2p}(W)$. We also have $u_t^{-1} \in \mathbb{D}_2^{2p}(W)$. Indeed, since the initial value u_0 is bounded from below by $\varepsilon_0 > 0$, we have $u_t \geq \varepsilon_0$ for all $t \geq 0$. From $\nabla(u_t^{-1}) = -u_t^{-2} \nabla u_t$ it follows that $|\nabla(u_t^{-1})|_H \leq \varepsilon_0^{-2} |\nabla u_t|_H$. Thus $\nabla(u_t^{-1}) \in L^{4p}(W, H)$. Next

$$\nabla^2(u_t^{-1}) = 2u_t^{-3} \nabla u_t \otimes \nabla u_t - u_t^{-2} \nabla^2 u_t,$$

hence

$$\|\nabla^2(u_t^{-1})\|_{H \otimes H} \leq 2\varepsilon_0^{-3} |\nabla u_t|_H^2 + \varepsilon_0^{-2} \|\nabla^2 u_t\|_{H \otimes H}$$

which implies that $\nabla^2(u_t^{-1}) \in L^{2p}(W, H \otimes H)$. To sum up, $u_t^{-1} \in \mathbb{D}_2^{2p}(W)$. Therefore by [12, Proposition 1.5.6], we see that $\frac{|\nabla u_t|_H^2}{u_t} \in \mathbb{D}_2^p(W)$.

Now

$$\mathcal{L} \left(\frac{|\nabla u_t|_H^2}{u_t} \right) = u_t^{-1} \mathcal{L}(|\nabla u_t|_H^2) + |\nabla u_t|_H^2 \mathcal{L}(u_t^{-1}) + 2 \langle \nabla(u_t^{-1}), \nabla(|\nabla u_t|_H^2) \rangle_H. \quad (3.10)$$

By the Weitzenböck formula proved in Theorem 2.2,

$$\mathcal{L}(|\nabla u_t|_H^2) = 2 \langle \nabla u_t, \nabla \mathcal{L} u_t \rangle_H + 2 |\nabla u_t|_H^2 + 2 \|\nabla^2 u_t\|_{H \otimes H}^2. \quad (3.11)$$

It is easy to show that $\mathcal{L}(u_t^{-1}) = -u_t^{-2} \mathcal{L} u_t + 2u_t^{-3} |\nabla u_t|_H^2$ and

$$\langle \nabla(u_t^{-1}), \nabla(|\nabla u_t|_H^2) \rangle_H = -2u_t^{-2} \langle \nabla^2 u_t, \nabla u_t \otimes \nabla u_t \rangle_{H \otimes H}.$$

Substituting these equalities and (3.11) into (3.10) gives us

$$\mathcal{L} \left(\frac{|\nabla u_t|_H^2}{u_t} \right) = \frac{2}{u_t} \langle \nabla u_t, \nabla \mathcal{L} u_t \rangle_H - \frac{|\nabla u_t|_H^2}{u_t^2} \mathcal{L} u_t + \frac{2}{u_t} |\nabla u_t|_H^2 + \frac{2}{u_t} \left\| \nabla^2 u_t - \frac{\nabla u_t \otimes \nabla u_t}{u_t} \right\|_{H \otimes H}^2.$$

Next it is clear that

$$\frac{\partial}{\partial t} \left(\frac{|\nabla u_t|_H^2}{u_t} \right) = \frac{2}{u_t} \langle \nabla u_t, \nabla \left(\frac{\partial}{\partial t} u_t \right) \rangle_H - \frac{|\nabla u_t|_H^2}{u_t^2} \frac{\partial}{\partial t} u_t. \quad (3.12)$$

Combining the above two equalities, we obtain the desired formula. \square

Finally we are in the position to prove the main result of this paper.

Proof of Theorem 1.1. By the definition of entropy,

$$\frac{d}{dt} \text{Ent}(u_t) = - \frac{d}{dt} \int_W u_t \log u_t \, d\mu.$$

To commute the differential and integral, we have to check the conditions in Theorem 4.1 for the function $(0, \infty) \times W \ni (t, w) \mapsto u_t(w) \log u_t(w)$. The first condition is obviously satisfied. In view of Proposition 3.3, we have

$$\frac{\partial}{\partial t}(u_t \log u_t) = (\log u_t) \mathcal{L}u_t + \mathcal{L}u_t,$$

hence the condition (ii) is verified. It remains to check condition (iii). Let $0 < a < b < \infty$. Taking $m = 0$ in (3.6), we obtain that

$$\|\mathcal{L}u_t\|_{L^{2p}(W)} \leq \tilde{C}_p \|u_0\|_{L^{2p}} \sum_{i=0}^2 A_t^i, \quad t > 0. \quad (3.13)$$

Since $u_t \geq \varepsilon_0 > 0$, it is clear that $\log \varepsilon_0 \leq \log u_t \leq u_t$, hence

$$|\log u_t| \leq |\log \varepsilon_0| \vee u_t \leq |\log \varepsilon_0| + u_t.$$

Therefore by the contraction property of the Ornstein-Uhlenbeck semigroup P_t ,

$$\sup_{t>0} \|\log u_t\|_{L^{2p}} \leq |\log \varepsilon_0| + \sup_{t>0} \|u_t\|_{L^{2p}} \leq |\log \varepsilon_0| + \|u_0\|_{L^{2p}}. \quad (3.14)$$

Now by (3.13) and (3.14), Cauchy's inequality gives us

$$\int_a^b \int_W |(\log u_t) \mathcal{L}u_t| \, d\mu dt \leq \int_a^b \|(\log u_t) \mathcal{L}u_t\|_{L^p} \, dt \leq \int_a^b \|\log u_t\|_{L^{2p}} \|\mathcal{L}u_t\|_{L^{2p}} \, dt < +\infty.$$

Hence the condition (iii) in Theorem 4.1 is satisfied too. We pass the differentiation into the integral sign and get

$$\frac{d}{dt} \text{Ent}(u_t) = - \int_W [(\log u_t) \mathcal{L}u_t + \mathcal{L}u_t] \, d\mu.$$

We have

$$|\nabla \log u_t|_H = |u_t^{-1} \nabla u_t|_H \leq \varepsilon_0^{-1} |\nabla u_t|_H,$$

hence $\log u_t \in \mathbb{D}_1^{2p}(W)$. By the integration by parts formula we get

$$\frac{d}{dt} \text{Ent}(u_t) = - \int_W (\log u_t) \mathcal{L}u_t \, d\mu = \int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu. \quad (3.15)$$

Next we have $\int_W \mathcal{L} \left(\frac{|\nabla u_t|_H^2}{u_t} \right) \, d\mu = 0$, again due to the integration by parts formula. By (3.12), we can prove in a similar way that the function $(t, w) \mapsto \frac{|\nabla u_t|_H^2}{u_t}(w)$ satisfies the three conditions in Theorem 4.1, hence

$$\frac{d}{dt} \int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu = \int_W \frac{\partial}{\partial t} \left(\frac{|\nabla u_t|_H^2}{u_t} \right) \, d\mu.$$

Therefore, integrating both sides of the formula in Theorem 3.4, we obtain

$$\begin{aligned} -\frac{d}{dt} \int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu &= 2 \int_W \frac{1}{u_t} \left| \nabla^2 u_t - \frac{\nabla u_t \otimes \nabla u_t}{u_t} \right|_{H \otimes H}^2 \, d\mu + 2 \int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu \\ &\geq 2 \int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu. \end{aligned}$$

Using Proposition 2.1(2) (it is also true for functionals in $L^p(W, H)$), we can show that $t \mapsto \int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu$ is right continuous at $t = 0$. Therefore

$$\int_W \frac{|\nabla u_t|_H^2}{u_t} \, d\mu \leq e^{-2t} \int_W \frac{|\nabla u_0|_H^2}{u_0} \, d\mu.$$

In view of (3.15), the proof is complete. \square

4 Appendix: a result on the differentiation under the integral sign

In the proof of Theorem 1.1, we need the following theorem from the analysis which guarantees the differentiation under the integral sign. See [1] for an introduction of related results. For the reader's convenience, we give its complete proof here.

Theorem 4.1. *Let T be an open interval of \mathbb{R} , and (Ω, ν) a measure space. Suppose that a function $f : T \times \Omega \rightarrow \mathbb{R}$ satisfies the following conditions:*

- (i) *$f(t, \omega)$ is a measurable function of t and ω jointly, and is integrable over Ω for almost all $t \in T$ fixed;*
- (ii) *for almost all ω , the derivative $\frac{\partial}{\partial t} f(t, \omega)$ exists for all $t \in T$;*
- (iii) *for all compact intervals $[a, b] \subset T$, we have*

$$\int_a^b \int_{\Omega} \left| \frac{\partial}{\partial t} f(t, \omega) \right| d\nu(\omega) dt < +\infty.$$

Then for a.e. $t \in T$,

$$\frac{d}{dt} \int_{\Omega} f(t, \omega) d\nu(\omega) = \int_{\Omega} \frac{\partial}{\partial t} f(t, \omega) d\nu(\omega).$$

Proof. For $t \in T$, define $F(t) = \int_{\Omega} f(t, \omega) d\nu(\omega)$. Then $t \mapsto F(t)$ is absolutely continuous on T . Indeed, for any $\varepsilon > 0$, we conclude from condition (iii) that there is $\kappa > 0$, such that for any measurable set $E \subset [a, b]$ whose Lebesgue measure is less than κ , it holds

$$\int_E \int_{\Omega} \left| \frac{\partial}{\partial t} f(t, \omega) \right| d\nu(\omega) dt < \varepsilon.$$

For any finite sequence of pairwise disjoint intervals (x_k, y_k) of T satisfying $\cup_k (x_k, y_k) \subset [a, b]$ and $\sum_k (y_k - x_k) < \kappa$, we have

$$\sum_k |F(y_k) - F(x_k)| \leq \sum_k \int_{\Omega} |f(y_k, \omega) - f(x_k, \omega)| d\nu(\omega). \quad (4.1)$$

By (iii) and Fubini's theorem, for any $0 < a < b < \infty$,

$$\int_{\Omega} \int_a^b \left| \frac{\partial}{\partial t} f(t, \omega) \right| ds d\nu(\omega) < \infty,$$

hence for ν -a.e. $\omega \in \Omega$, $\int_a^b \left| \frac{\partial}{\partial t} f(t, \omega) \right| dt < \infty$. As a result,

$$f(b, \omega) - f(a, \omega) = \int_a^b \frac{\partial}{\partial t} f(t, \omega) dt.$$

By (4.1) and Fubini's theorem, we obtain

$$\begin{aligned} \sum_k |F(y_k) - F(x_k)| &\leq \sum_k \int_{\Omega} \int_{x_k}^{y_k} \left| \frac{\partial}{\partial t} f(t, \omega) \right| ds d\nu(\omega) \\ &= \int_{\cup_k (x_k, y_k)} \int_{\Omega} \left| \frac{\partial}{\partial t} f(t, \omega) \right| d\nu(\omega) ds < \varepsilon. \end{aligned}$$

Therefore F is locally absolutely continuous, and $\frac{d}{dt}F(t)$ exists for a.e. $t \in T$.

Next for $t \in T$, define $g(t) = \int_{\Omega} \frac{\partial}{\partial t} f(t, \omega) d\nu(\omega)$. By assumption (iii), the function g is locally integrable on T . Fix $a \in T$, define $G(t) = \int_a^t g(s) ds$ for all $t \in T$. Then by Lebesgue's differentiation theorem, we have

$$g(t) = \frac{d}{dt}G(t) \quad \text{for a.e. } t \in T. \quad (4.2)$$

Next for a.e. $t \in T$,

$$\frac{d}{dt}G(t) = \lim_{h \downarrow 0} \frac{1}{h} (G(t+h) - G(t)) = \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} \int_{\Omega} \frac{\partial}{\partial s} f(s, \omega) d\nu(\omega) ds.$$

Due to condition (iii), we can apply Fubini's theorem to get

$$\begin{aligned} \frac{d}{dt}G(t) &= \lim_{h \downarrow 0} \frac{1}{h} \int_{\Omega} \int_t^{t+h} \frac{\partial}{\partial s} f(s, \omega) ds d\nu(\omega) \\ &= \lim_{h \downarrow 0} \frac{1}{h} \int_{\Omega} (f(t+h, \omega) - f(t, \omega)) d\nu(\omega) = \frac{d}{dt}F(t). \end{aligned}$$

Combining this with (4.2) and the definitions of F, g , we complete the proof. \square

References

- [1] S. Cheng, *Differentiation under the integral sign*, <http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html#tex2html13>.
- [2] S. Fang, *Introduction to Malliavin calculus*. Math. Series for Graduate students, vol. 3, 2005, Tsinghua University Press, Springer.
- [3] S. Fang, J. Shao and K.T. Sturm, *Wasserstein space over the Wiener space*. Probab. Theory Relat. Fields 146 (2010), no. 3-4, 535–565.
- [4] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*. North-Holland, Amsterdam, 1989.
- [5] C. LeBris and P.L. Lions, *Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients*. Comm. Partial Differential Equations 33 (2008), 1272–1317.
- [6] Adrian P. C. Lim and Dejun Luo, *Asymptotic estimates on the time derivative of entropy on a Riemannian manifold*. arXiv:1011.3979.
- [7] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*. Ann. of Math. 169 (2009), no. 3, 903–991.
- [8] Dejun Luo, *Well-posedness of Fokker-Planck type equations on the Wiener space*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13 (2010), no. 2, 273–304.
- [9] P. Malliavin, *Stochastic analysis*, Grund. Math. Wissen., vol. 313, Springer, 1997.
- [10] Lei Ni, *The entropy formula for linear heat equation*. J. Geom. Anal. 14 (2004), no. 1, 87–100.

- [11] Lei Ni, *Addenda to “The entropy formula for linear heat equation”*. J. Geom. Anal. 14 (2004), no. 2, 369–374.
- [12] D. Nualart, *The Malliavin calculus and related topics*. Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006.
- [13] I. Shigekawa, *The de Rham-Hodge-Kodaira decomposition on abstract Wiener space*. J. Math. Kyoto Univ. 26 (1986), 191–202.
- [14] K.Th. Sturm, *On the geometry of measures spaces. I*. Acta Math. 196 (2006), 65–131.
- [15] K.Th. Sturm, *On the geometry of measures spaces. II*. Acta Math. 196 (2006), 133–177.
- [16] H. Sugita, *Positive generalized Wiener functionals and potential theory over abstract Wiener space*. Osaka J. Math. 25 (1988), 665–696.